Last time:
K field

Def: A norm (or absolute value) on K is a fct. $|\cdot|: K \to \mathbb{R}_{\geq 0}$, s.t.

1) $|x| = 0$ iff $x = 0$
2) $|x \cdot y| = |x| \cdot |y| \quad \forall x, y \in K$
3) $|x + y| \leq |x| + |y| \quad \forall x, y \in K$

triangle inequality

usual abs. val.

Ex: $(\mathbb{Q}_p, |\cdot|_p)$, $(\mathbb{R}, |\cdot|_\infty)$, $(\mathbb{C}, |\cdot|_\infty)$

p-adic norm

$|\cdot|_p$ satisfies ultrametric $\triangle$-ineq.

3') $|x + y| \leq \max\{|x|, |y|\} \quad \forall x, y \in K$

Norms satisfying 3') are called non-archimedean

If $|\cdot|$ is not non-arch, then

$|-|$ is called archimedean

If $|-|$ is non-arch, then

$$|-| = q^{-v(-)}, \quad q \in \mathbb{R},\ q > 1$$

$v : K \to \mathbb{R} \cup \{\infty\}$ add. valuation, i.e.

1) $v(x) = \infty$ iff $x = 0$
2) $v(x \cdot y) = v(x) + v(y) \quad \forall x, y \in K$
3) $v(x + y) \geq \min(v(x), v(y)) \quad \forall x, y \in K$

$$|-| \rightsquigarrow v(-) := -\log_q(|-|)$$

$$|-| := q^{-v(x)} \leftarrow v(-)$$

preserve equiv. of norms resp. add. valuations.

Def: A discretely valued field K is non-arch. valued field

$(K, |-|)$, s.t. $|K^\times| \leq \mathbb{R}_{>0}$ is a discrete subgrp.

$(\Leftrightarrow -\log_q |K^\times| = v(K^\times) \leq \mathbb{R}$ is a discrete subgroup)

$(\Leftrightarrow v(K^\times) = \mathbb{Z} \cdot a$ for some $a \in \mathbb{R}_{>0}$)

Ex: $\cdot$ $(\mathbb{Q}_p, |-|_p)$ is discretely valued

$\cdot$ k any field,

$k[[z]]$, $K = \mathrm{Frac}(k[[z]])$

$$\mathrm{Frac}(k[[z]]) = k((z)).$$

$$v : k((z)) \to \mathbb{Z} \cup \{\infty\}$$

$$x = \sum_{n \gg -\infty}^{\infty} a_n z^n \mapsto \begin{cases} \infty , & x = 0 \\ \min(n \mid a_n \neq 0), & \text{otherwise} \end{cases}$$

$a_n \in k$

$\Rightarrow v$ add. val.
$\sim$
discrete

Usually discrete valuations are normalized, i.e. $v(K^\times) = \mathbb{Z}$
(in the above $a = 1$)

Ex: 1) K any number field,
$\sigma : K \hookrightarrow \mathbb{C}$
$\Rightarrow |x|_\sigma := |\sigma(x)|_{\mathbb{C}}$ defines archimedean norm on K
(we'll see for σ not conj. to σ', these norms are inequivalent & they exhaust all equiv. classes of arch. norms on K)
We get

$$\hat{K}_{\xi} = \begin{cases} \mathbb{R} & , \sigma(K) \subseteq \mathbb{R} \\ \mathbb{C} & , \sigma(K) \nsubseteq \mathbb{R} \end{cases}$$

for the top. induced by $|-|_\sigma$

Let $\mathfrak{p} \subseteq \mathcal{O}_K$ be any maximal ideal

Recall

$$v_{\mathfrak{p}} : K \to \mathbb{Z} \cup \{\infty\},$$

$$x \mapsto \begin{cases} \infty & , x = 0 \\ a & \text{if } x \cdot \mathcal{O}_K = \mathfrak{p}^a \cdot \frac{\mathfrak{m}}{\mathfrak{n}} \end{cases}$$

$\mathfrak{m}, \mathfrak{n} \subseteq \mathcal{O}_K$ ideals prime to $\mathfrak{p}$

$\Rightarrow v_{\mathfrak{p}}$ is an additive valuation

$K_{\mathfrak{p}}$:= completion of K for

$$|-|_{\mathfrak{p}} = N(\mathfrak{p})^{-v_{\mathfrak{p}}(-)}$$

"normalized $\mathfrak{p}$-adic norm"

(here we put $q = N(\mathfrak{p}) > 1$)

Later, $K_{\mathfrak{p}}$ finite ext of $\mathbb{Q}_p$,

$(p) = \mathbb{Z} \cap \mathfrak{p}$

& these exhaust all equiv. classes of non-arch. norms on K

Note: Definition of $v_{\mathfrak{p}}$ works for any Dedekind ring A and any maximal ideal $\mathfrak{p} \subseteq A$,

e.g. k any field, $A = k[z]$,

$\mathfrak{p} = (z)$

The $\mathfrak{p}$-adic completion on $K := \mathrm{Frac}(A) = k(z)$ is

$k((z))$

Prop: K field, $|\cdot|$ norm on K.
Then $|\cdot|$ is non-arch.
if and only if $|\cdot|$ is bounded on the image of $\mathbb{Z}$ in K.
In part, if char $K > 0$ each norm on K is non-arch.

Prf: "$\Rightarrow$" $|\cdot|$ non-arch.

$$\underset{n \in \mathbb{Z}}{\Rightarrow} |n| = |1 + \ldots + 1| \le \underbrace{\max(|1|)}_{=1}$$

"$\Leftarrow$" Assume $|n| \le C \;\; \forall n \in \mathbb{Z}$

Let $x \in K$, $|x| \le 1$

$$\Rightarrow |(1+x)^n| = \left| \sum_{i=0}^{n} \binom{n}{i} x^i \right| \le C \cdot (n+1)$$

$$\Rightarrow |(1+x)| \le 1$$

$$\left(\text{as } \sqrt[n]{C(n+1)} \to 1, \; n \to \infty\right)$$

Now, let $x, y \in K$, $|x| \geq |y|$, $x \neq 0$

$$\Rightarrow |x+y| = |x| \cdot \underbrace{\left|1 + \frac{y}{x}\right|}_{\leq 1} \leq |x| = \max\{|x|, |y|\}$$

$\Rightarrow |-|$ is non-arch $\square$

La (strong triangle inequality)

⚠ $(K, |-|)$ non-arch. valued field, $x, y \in K$

1) If $|x| \neq |y|$, then

$$|x+y| = \max(|x|, |y|)$$

2) If $|x+y| < |x|$, then $|x| = |y|$

In particular, if $x_n \to x$, $n \to \infty$ in K & $x \neq 0$

Then $|x_n| = |x|$ for $n \gg 0$

Indeed, let $0 < \varepsilon < |x|$ and n_0, s.t. $|x_n - x| < \varepsilon \;\forall n \geq n_0$

$\Rightarrow |x - x_n| < \varepsilon < |x| \overset{2)}{\Rightarrow} |x_n| = |x|$

Proof (of (a)): 1) Assume $|x| > |y|$

$\Rightarrow$

$$|y| < |x| = |x + y - y| \leq \max(|x+y|, |y|) = |x+y|$$

$$\Rightarrow |x| \leq |x+y| \leq \max(|x|, |y|) = |x|$$

2) follows from 1) $\square$

Def: $(K, |\cdot|)$ non-arch. valued field

$$\Rightarrow \mathcal{O}_K := \{x \in K \mid |x| \leq 1\}$$

$$\overset{?}{=} \{x \in K \mid v(x) \geq 0\}$$

$$|\cdot| = q^{-v(\cdot)}, \; q > 1$$

valuation ring of $(K,|-|)$

implied by $|x+y| \leq \max(|x|,|y|) \quad \forall x,y \in K$

For each $r \in \mathbb{R}_{\geq 0}$

$\Rightarrow \{x \in K \mid |x| \leq r\}, \{x \in K \mid |x| < r\}$

are $\mathcal{O}_K$-submodules of K

(in particular, ideals in $\mathcal{O}_K$ if $1 \geq r$)

If $|-|$ discrete, then $\mathcal{O}_K$ is called a discrete valuation ring.

Prop: $(K,|-|)$ non-arch. valued field, $\mathcal{O}_K \subseteq K$

1) $\mathcal{O}_K$ integrally closed, local

with maximal ideal

$\mathfrak{m}_K = \{ x \in K \mid |x| < 1 \}$.

each fin. gen. ideal $I \subseteq \mathcal{O}_K$ is principal and of the form $\{ x \in K \mid |x| \leq r \}$ for some $r \geq 0$ (actually $r = |y|$, if $(y) = I$)

(for the metric top. on K)

2) The subspace top. on $\mathcal{O}_K$ is (x)-adic for each $x \in \mathfrak{m}_K \setminus \{0\}$ (If no such x exists, set $(x) := \{0\}$), $\leftarrow$ happens iff $|\cdot|$ trivial

$\mathcal{O}_K \subseteq K$ is open + closed

$K = \mathcal{O}_K[\frac{1}{x}]$ & $\hat{K} = \hat{\mathcal{O}}_K[\frac{1}{x}]$

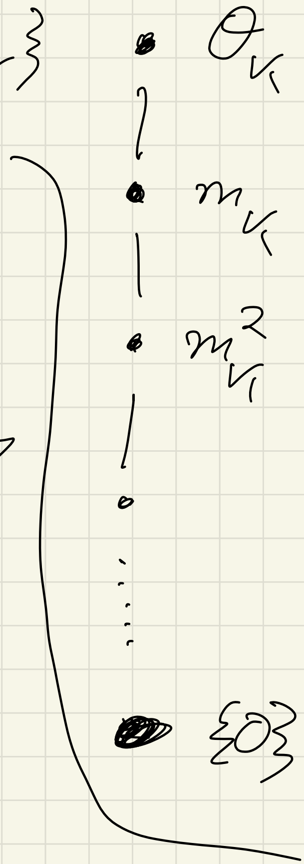
(x)-adic completion

3) If $|\cdot|$ discrete, $\mathfrak{m}_K$ is principal

$$\{ I \subseteq \mathcal{O}_K \text{ non-zero} \} \xleftrightarrow{1:1} \mathbb{N}_{\geq 0}$$

$$\mathfrak{m}_K^n \longmapsfrom n$$

and $\operatorname{Spec} \mathcal{O}_K = \{ (0), \mathfrak{m}_K \}$



In particular, $\mathcal{O}_K$ is a local principal ideal domain. &

$$\{ \text{non-zero fract. ideals} \} \xleftrightarrow{1:1} \mathbb{Z}$$

$$\mathfrak{m}_K^n \longmapsfrom n$$

Proof: 1) Let $x \in K$ be integral over $\mathcal{O}_K$.

$$x^n + a_1 x^{n-1} + \dots + a_n = 0,$$

$$a_1, \dots, a_n \in \mathcal{O}_K$$

If $|x| > 1 \Rightarrow |x|^n > |x^{n-i}| \geq |a_i \cdot x^{n-i}|$

$\forall\, i \geq 1$

$$\underset{\text{strong } \triangle}{\Rightarrow} \underbrace{|x^n + a_1 x^{n-1} + \dots + a_n|}_{=\,0} = |x^n| > 1 \quad ↯$$

$$\Rightarrow x \in \mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$$

If $x \in \mathcal{O}_K \setminus m_K \Rightarrow |x| = 1$

$$\Rightarrow |\underset{K}{x^{-1}}| = 1 \Rightarrow x^{-1} \in \mathcal{O}_K$$

$\Rightarrow \mathcal{O}_K$ local with max. ideal

m_K

Assume $I = (f_1, \dots, f_n)$

wlog $\underset{\underset{0}{\neq}}{r} := |f_1| \geq |f_i| \quad \forall i = 1, \dots, n$

Claim: $I = (f_1) = \{x \in K \mid |x| \leq r\}$

Clear $(f_1) \subseteq I \underset{\nearrow}{\subseteq} \{x \in K \mid |x| \leq r\}$

ultrametric Δ ineq.

Pick $y \in K, |y| \leq r$

$\Rightarrow \left|\frac{y}{f_1}\right| \leq 1$, i.e. $\frac{y}{f_1} \in O_K$

$\Rightarrow y = \underbrace{\frac{y}{f_1}}_{\in O_K} \cdot f_1 \in (f_1) = O_K \cdot f_1$

$\Rightarrow$ 1) proven

2) $\mathcal{O}_{K_1} \leq K$ is open (+ closed)

Namely, set $U_r = \{x \in K \mid |x| < r\}$

$\Rightarrow U_r \leq K$ open subgroup

& $\{U_r\}_{r>0}$ forms a fund. syst. of open nbhds of 0

Set $Z_r := \{x \in K \mid |x| \leq r\}$

$$\Rightarrow U_r \subseteq Z_r = \coprod_{a \in Z_r/U_r} a + U_r \quad \text{open}$$

$Z_r, U_r \leq K$ subgroups

If $\varepsilon > 0$

$\Rightarrow U_r \subseteq Z_r \subseteq U_{r+\varepsilon}$

$\Rightarrow \{Z_r\}_{r>0}$ form a fund system of open nbds of 0

In part, $\mathcal{O}_K = Z_1$ is open

Pick $x \in \mathfrak{m}_K \setminus \{0\}$, $s := |x|$

$\Rightarrow 0 < s < 1$ &

$\{Z_{s^n}\}_{n \geq 0}$ is a fund. system of open nbhds of 0

By the proof of 1)

$$Z_{s^n} = x^n \cdot \mathcal{O}_K$$

$\Rightarrow$ the subspace top. on $\mathcal{O}_K$ is the (x)-adic top.

$K = \mathcal{O}_K[\frac{1}{x}]$ is clear:

$\mathcal{O}_K[\frac{1}{x}] \subseteq K$ and if $y \in K$

$\Rightarrow \exists\, n \geq 0$, s.t. $|x|^n \cdot |y| \leq 1$

(x is top. nilpotent, i.e. $x^m \to 0$, $m \to \infty$, equiv. $|x|^m \to 0,\ m \to \infty$)

$$\Rightarrow \quad y = \frac{y \cdot x^n}{x^n} \in \mathcal{O}_K\left[\frac{1}{x}\right]$$

Consider $\mathcal{O}_{\hat{K}} = \{x \in \hat{K} \mid |x|_{\hat{K}} \leq 1\}$,

$x \in m_K \setminus \{0\}$

$\uparrow$ not $\hat{K}$

$\Rightarrow$ $\mathcal{O}_{\hat{K}}$ (x)-adic. complete

$\mathcal{O}_K$ - alg., i.e. $\mathcal{O}_{\hat{K}} = \varprojlim_n \mathcal{O}_{\hat{K}}/x^n\mathcal{O}_{\hat{K}}$

$\Rightarrow$ ex. canonical morphism

$$\underbrace{\widehat{\mathcal{O}}_{K,(x)}}_{(x)\text{-adic compl. of } \mathcal{O}_K} \to \mathcal{O}_{\hat{K}}$$

R I-adic, $I = (r)$, r non-zero div.

$\Rightarrow \hat{R}/r^n\hat{R} = R/r^nR$

Suff. to proof:

$$\mathcal{O}_K/(x^n) = \widehat{\mathcal{O}}_{K,(x)}/(x^n) \simeq \mathcal{O}_{\hat{K}}/x^n\mathcal{O}_{\hat{K}}$$

$\forall n \geq 0$

wlog $n = 1$ (replace x by x^n)

Claim: $(K, |\cdot|)$ non-arch. valued,
$x \in m_K \setminus \{0\}$ (x is top. nilp.)

$\Rightarrow \mathcal{O}_K/(x) \cong \mathcal{O}_{\hat{K}}/(x)$

Proof next time.

$(\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}])$

$\mathrm{Frac}(\mathbb{Z}_p)$